

# Time Consistent G-Expectation and Bid-Ask Dynamic Pricing Mechanisms for Contingent Claims Under Uncertainty \*

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**Abstract** We study dynamic pricing mechanisms of European contingent claims under uncertainty by using G framework introduced by Peng ([21]). We consider a financial market consists of a riskless asset and a risky stock with price process modelled by a geometric generalized G-Brownian motion, which features the drift uncertainty and volatility uncertainty of the stock price process. A time consistent G-expectation is defined by the viscosity solution of the G-heat equation. Using the time consistent G-expectation we define the G dynamic pricing mechanism for the claim. We prove that G dynamic pricing mechanism is the bid-ask Markovian dynamic pricing mechanism. The full nonlinear PDE is derived to describe the bid (resp. ask) price dynamic of the claims. Monotone characteristic finite difference schemes for the nonlinear PDE are given, and the simulations of the bid (resp. ask) prices of contingent claims with uncertainty are implemented.

**Keywords** G Brownian motion, G expectation, uncertainty model, dynamic pricing mechanism, monotone finite difference

## 1 Introduction

In probability framework, the uncertain model of the stock price is assume that the stock price be a positive stochastic process that satisfies the generalized geometric Brownian motion

$$d \log \tilde{S}_t = \mu_t^p dt + \sigma_t^p dW_t, \quad (1.1)$$

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where  $(W_t)_{t \geq 0}$  be a 1-dimensional standard Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$  and  $(\mathcal{F}_t : 0 \leq t \leq T)$  is the filtration generated by  $W_t$ , and  $(\mu_t^P, \sigma_t^P)_{t \geq 0}$  is unknown such that

$$(\mu_t^P, \sigma_t^P) \in [\underline{\mu}, \bar{\mu}] \times [\underline{\sigma}, \bar{\sigma}], \quad \underline{\mu}, \bar{\mu}, \underline{\sigma}, \bar{\sigma} \text{ are constants such that } \underline{\mu} \leq \bar{\mu}, \underline{\sigma} \leq \bar{\sigma}. \quad (1.2)$$

We denote  $\Gamma$  as all possible paths  $(\mu_t^P, \sigma_t^P)_{t \geq 0}$  satisfying (1.2), then  $\Gamma$  is a closed convex set. For each fixed path  $(\gamma_t)_{t \geq 0} = (\mu_t^P, \sigma_t^P)_{t \geq 0} \in \Gamma$ , let  $P_\gamma$  be the probability measure on the space of continuous paths  $(C(0, \infty), \mathcal{B}(C(0, \infty)))$  induced by  $\int_0^t (\mu_s^P ds + \sigma_s^P dW_s)$ , and denote  $P_0$  as the reference probability measure induced by  $W_t$ . We set  $\mathcal{P}$  be the class of all such probability measures  $P_\gamma$ , and for each  $P \in \mathcal{P}$  we denote  $E_P$  the corresponding expectation.

If the uncertainty comes from  $\mu_t^P$  which is called drift uncertainty, each probability  $P \in \mathcal{P}$  is absolutely continuous with the reference measure  $P_0$ . Chen and Epstein [7] proposed to use g-expectation introduced in [18] for a robust valuation of stochastic utility under drift uncertainty. Karoui, Peng and Quenez in [15] and Peng in [19] proposed to use time consistent condition g-expectation  $E_{t,T}^g : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t)$  defined by solutions of a BSDE, as bid-ask dynamic pricing mechanism for European contingent claims  $\xi$

$$\text{ask price: } E_{t,T}^g[\tilde{H}_T^t \xi] = \sup_{P \in \mathcal{P}} E_P[\tilde{H}_T^t \xi | \mathcal{F}_t], \quad \text{bid price: } -E_{t,T}^g[-\tilde{H}_T^t \xi] = -\sup_{P \in \mathcal{P}} E_P[-\tilde{H}_T^t \xi | \mathcal{F}_t],$$

where  $(\tilde{H}_s^t; s \geq t)$  be some deflator. Delbaen, Peng and Rosazza ([8]) proved that any coherent and time consistent risk measure absolutely continuous with respect to the reference probability can be approximated by a g-expectation.

The volatility uncertainty model was initially studied by Avellaneda, Levy and Paras [2] and Lyons [14] in the risk neutral probability measure, they intuitively give the bid-ask price of a European contingent claim as follows

$$\text{ask price: } \sup_{Q \in \mathcal{Q}} E_Q[e^{-r(T-t)} \xi | \mathcal{F}_t], \quad \text{bid price: } -\sup_{Q \in \mathcal{Q}} E_Q[e^{-r(T-t)} \xi | \mathcal{F}_t]. \quad (1.3)$$

where  $\mathcal{Q} = \{Q : Q \text{ is the risk neutral probability measure of } P, \forall P \in \mathcal{P}\}$ ,  $r$  is the short interest rate. For the volatility uncertainty, the probabilities in  $\mathcal{P}$  are mutually singular, the upper-expectation can not be approximated by any g-expectation.

Motivated by the problem of coherent risk measures under the volatility uncertainty ([1]), Peng ([21],[22]) introduced a sublinear expectation on a well defined sublinear expectation space, under which the canonical process  $(B_t)_{t \geq 0}$  is defined as G-Brownian. The increments of the G-Brownian motion are zero-mean, independent and stationary and  $N(\{0\}, [\underline{\sigma}^2 t, \bar{\sigma}^2 t])$ , and the corresponding sublinear expectation is called G-expectation. By using quasi-sure stochastic analysis, Denis, Hu and Peng in [11] and Sonar, Touzi and Zhang in [24] constructed consistent G-expectation, condition G-expectation and G-Brownian motion such that the condition G-expectation can be represented as an upper condition expectation

$$E_t^G[\xi] = \text{esssup}_{Q \in \mathcal{Q}} E_Q[\xi | \mathcal{F}_t], \quad \xi \in L_G^1, \quad Q - a.s. \quad (1.4)$$

and  $M_t$  is a G-martingale if and only if

$$M_s = \text{esssup}_{Q \in \mathcal{Q}} E_Q[M_t | \mathcal{F}_s], \quad 0 < s < t, \quad Q - a.s. \quad (1.5)$$

which means that the condition G-expectation could be used as bid-ask dynamic pricing mechanism under volatility uncertainty, and a G-martingale is a supermartingale for each risk neutral probability measure space.

There is a literature on the pricing the contingent claims under volatility uncertainty recently, Denis and Martin [12] studied the superhedging of the claims by using quasi-sure analysis, and

Vorbink [27] derived the lower arbitrage price and upper arbitrage price based on the approach of arbitrage by using G-framework. For more general situation, if the uncertainty comes from drift and volatility coefficients, Peng [20] studied the super evaluation of the contingent claims by using filtration consistent nonlinear expectations theory.

In this paper we study the dynamic pricing mechanisms of European contingent claims written on a risky asset under uncertainty by using G-framework introduced by Peng (2005, [21]). At first, in a path sublinear space  $(\Omega, L_{ip}(\Omega), \hat{E})$  we model the price process of the risky asset by a generalized G-Brownian which describes the drift uncertainty and the volatility uncertainty of the stock. Using the techniques in G-framework we show that the risk premium of the stock is maximum distributed. We define the bid-ask dynamic price of the claim by using BSDE and derive a bid-ask dynamic pricing formula by using G-martingale representation theorem [25]. Further more we define a time consistent G-expectation  $E^G$  by the G-heat equation and define a corresponding G-Brownian. By the G-expectation  $E^G$  transform, the uncertainty model is transferred to volatility uncertainty model in  $(\Omega, L_{ip}(\Omega), E^G)$ , we prove that the condition G-expectation  $E_{t,T}^G$  is the bid-ask dynamic pricing mechanism for the claims. we also show that the bid-ask dynamic pricing mechanism  $E_{t,T}^G$  is a Markovian dynamic consistent pricing mechanism and characterize the bid (resp. ask) price by the viscosity solution of a full nonlinear PDE which is the Black-Scholes-Barenblatt equation intuitively given by Avellaneda, Levy and Paras([2]). For numerical computing the full nonlinear PDE, we propose monotone characteristic finite difference schemes for discrete solving the nonlinear PDE equations, provide iterative scheme for the discrete nonlinear system derived from the characteristic difference discretization, and analysis the convergence of the iterative solution to the viscosity solution of the nonlinear PDE. In the end, we give simulation examples for the ask and bid prices of contingent claims under uncertainty.

This paper is organized as follows. In Section 2, we give the financial market model. Section 3 we derive a bid-ask price formula for the European contingent claim. In Section 4, we propose a G-martingale transform and G dynamic pricing mechanisms for the claims. Section 5 we investigate the Markovian case for the G dynamic pricing mechanisms and full nonlinear PDEs are derived to describe the bid and ask prices for the claims. Numerical methods for the nonlinear PDEs are given in Section 6. The implementations of the simulations for the digital option and butterfly option under uncertainty are shown in Section 7.

## 2 The market model

We denote by  $\Omega = C(R^+)$  the space of all  $R$ -valued continuous paths  $(\omega_t)_{t \in R^+}$  with  $\omega_0 = 0$ , equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [(\max_{t \in [0, i]} |\omega_t^1 - \omega_t^2|) \wedge 1], \quad (2.1)$$

then  $(\Omega, \rho)$  is a complete separable metric space. For each fixed  $T \in [0, \infty)$ , we denote  $\Omega_T = \{\omega_{\cdot \wedge T} : \omega \in \Omega\}$ .

For the canonical process  $(B_t)(\omega) = \omega_t, t \in [0, \infty)$ , for  $\omega \in \Omega$ , we set

$$L_{ip}(\Omega_T) := \{\phi(B_{t_1}, \dots, B_{t_n}) : \forall n \in N, t_1, \dots, t_n \in [0, T], \forall \phi \in C_{b, Lip}(R^n)\},$$

and

$$L_{ip}(\Omega) := \cup_{n=1}^{\infty} L_{ip}(\Omega_n),$$

where  $C_{b, Lip}(R^n)$  denotes the linear space of functions  $\phi$  satisfying

$$|\phi(x) - \phi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \text{ for } x, y \in R, \\ \text{some } C > 0, m \in N \text{ depending on } \phi.$$

Assume that  $\underline{\mu}$ ,  $\bar{\mu}$ ,  $\underline{\sigma}$  and  $\bar{\sigma}$  are nonnegative constants such that  $\underline{\mu} \leq \bar{\mu}$  and  $\underline{\sigma} \leq \bar{\sigma}$ , we denote  $(\Omega, L_{ip}(\Omega), \hat{E})$  as a sublinear expectation space such that the canonical process  $(B_t)_{t \geq 0}$  is a generalized G-Brownian motion with

$$\begin{aligned} -\hat{E}[-B_t] &= \underline{\mu}t, & \hat{E}[B_t] &= \bar{\mu}t, \\ -\hat{E}[-B_t^2] &= \underline{\sigma}^2t, & \hat{E}[B_t^2] &= \bar{\sigma}^2t, \end{aligned} \quad (2.2)$$

and the generalized G-Brownian motion can be express as follows

$$B_t = \hat{B}_t + b_t,$$

where  $(\hat{B}_t)_{t \geq 0}$  is a G-Brownian motion and  $\hat{B}_t$  is  $N(\{0\}, [\underline{\sigma}^2t, \bar{\sigma}^2t])$  distributed, and  $b_t$  is  $N([\underline{\mu}t, \bar{\mu}t], \{0\})$  distributed. (Peng in [23] gave the construction of the sublinear expectation space  $(\Omega, L_{ip}(\Omega), \hat{E})$ , the generalized G-Brownian motion  $(B_t)_{t \geq 0}$ , G-Brownian motion  $(\hat{B}_t)_{t \geq 0}$  and  $N([\underline{\mu}t, \bar{\mu}t], \{0\})$  distributed  $(b_t)_{t \geq 0}$ .)

In this paper, we consider a financial market with a nonrisky asset (bond) and a risky asset (stock) continuously trading in market. The price  $P(t)$  of the bond is given by

$$dP(t) = rP(t)dt, \quad P(0) = 1, \quad (2.3)$$

where  $r$  is the short interest rate, we assume a constant nonnegative short interest rate. The stock price process  $S_t$  solves the following SDE

$$dS_t = S_t dB_t, \quad (\text{or } dS_t = S_t(db_t + d\hat{B}_t),) \quad (2.4)$$

where  $B_t$  is the generalized G-Brownian motion. The properties showed in (2.2) imply that the generalized G-Brownian describe the drift uncertainty and the volatility uncertainty of the stock price.

**Remark 1** The generalized G-Brownian motion can be characterized by the following nonlinear PDE:  $u(t, x) = \hat{E}[\varphi(x + B_t)]$  is the viscosity solution of

$$\partial_t u(t, x) - g(\partial_x u(t, x)) - G(\partial_{xx} u(t, x)) = 0, \quad u|_{t=0} = \varphi(x)$$

where  $\varphi(x)$  is a Lipschitz function,  $g(\alpha) = \bar{\mu}\alpha^+ - \underline{\mu}\alpha^-$  and  $G(\beta) = \frac{1}{2}(\bar{\sigma}^2\beta^+ - \underline{\sigma}^2\beta^-)$  for  $\alpha, \beta \in \mathbb{R}$ .

For investigation risk premium of the uncertainty model we give the representation of the process  $b_t$  which is  $N([\underline{\mu}t, \bar{\mu}t], \{0\})$  distributed as follows

**Lemma 2.1** For each fix  $t \in \mathbb{R}^+$ , we assume that  $\mu_t$  be identically distributed with  $b_1$ , i.e.,  $\mu_t$  be  $N([\underline{\mu}, \bar{\mu}], \{0\})$  distributed, and for  $t \geq 0$  we assume that  $\mu_t$  is independent from  $(\mu_{t_1}, \mu_{t_2}, \dots, \mu_{t_n})$  for each  $n \in \mathbb{N}$  and  $0 \leq t_1, \dots, t_n \leq t$ . Let  $\pi_t^N = \{t_0^N, t_1^N, \dots, t_N^N\}$  be a sequence of partitions of  $[0, t]$ , we define  $\int_0^t \mu_t dt = \sum_{k=0}^{N-1} \mu_k(t_{k+1}^N - t_k^N)$ , then  $\int_0^t \mu_t dt$  is  $N([\underline{\mu}t, \bar{\mu}t], \{0\})$  distributed in  $(\Omega, L_{ip}(\Omega), \hat{E})$ .

**Proof.** For  $t \geq 0$ , we denote  $a_t = \int_0^t \mu_t dt$ . For  $t, s \geq 0$  it is easy to check that  $a_{t+s} - a_t$  is identically distributed with  $a_s$  and independent from  $(a_{t_1}, a_{t_2}, \dots, a_{t_n})$  for each  $n \in \mathbb{N}$  and  $0 \leq t_1, \dots, t_n \leq t$ , and there holds

$$\hat{E}[a_t] = \bar{\mu}t, \quad -\hat{E}[-a_t] = \underline{\mu}t.$$

Since  $\mu_t$  is independent from  $(\mu_{t_1}, \mu_{t_2}, \dots, \mu_{t_n})$  for each  $n \in N$  and  $0 \leq t_1, \dots, t_n \leq t$ , we have

$$\begin{aligned}
\hat{E}[a_t^2] &= \hat{E}\left[\left(\sum_{k=0}^{N-1} \mu_k(t_{k+1}^N - t_k^N)\right)^2\right] \\
&= \hat{E}\left[\hat{E}\left[(y + \mu_{N-1}(t_N^N - t_{N-1}^N))^2 \mid y = \sum_{k=0}^{N-2} \mu_k(t_{k+1}^N - t_k^N)\right]\right] \\
&= \hat{E}\left[\left(\sum_{k=0}^{N-2} \mu_k(t_{k+1}^N - t_k^N) + \bar{\mu}(t_N^N - t_{N-1}^N)\right)^2\right] \\
&= \dots \\
&= \bar{\mu}^2 t^2.
\end{aligned} \tag{2.5}$$

For  $\varphi(x) \in C_{b.lib}(R)$  set  $u(t, x) = \hat{E}[\varphi(x + a_t)]$ , we have

$$|u(t, x) - u(t, y)| \leq C|x - y|, \quad \forall x, y \in R, \tag{2.6}$$

where  $C$  only depend on the Lipschitz constant, and in the rest of the proof  $C$  only depend on the Lipschitz constant,  $\underline{\mu}$  and  $\bar{\mu}$ .

For  $\delta \in [0, t]$ , since  $a_t - a_\delta$  is independent from  $a_\delta$ , we have

$$\begin{aligned}
u(t, x) &= \hat{E}[\varphi(x + a_\delta + a_t - a_\delta)] \\
&= \hat{E}[\hat{E}[\varphi(y + a_t - a_\delta) \mid y = x + a_\delta]] \\
&= \hat{E}[u(t - \delta, x + a_\delta)],
\end{aligned} \tag{2.7}$$

then from (2.5) and (2.6), we have

$$\begin{aligned}
&|u(t, x) - u(t - \delta, x)| \\
&\leq \hat{E}[|u(t - \delta, x + a_\delta) - u(t - \delta, x)|] \\
&\leq C\hat{E}[|a_\delta|] \leq C\delta.
\end{aligned} \tag{2.8}$$

Thus we prove that  $u(t, x)$  is Lipschitz in  $x$  and  $t$ . Consider the following PDE

$$\begin{aligned}
\partial_t u - g(\partial_x u) &= 0, \quad (t, x) \in R^+ \times R, \\
u|_{t=0} &= \varphi,
\end{aligned} \tag{2.9}$$

where

$$g(\alpha) = \bar{\mu}\alpha^+ - \underline{\mu}\alpha^-, \quad \forall \alpha \in R.$$

To prove  $u(t, x)$  is the viscosity solution of (2.9), for fixed  $(t, x) \in R^+ \times R$  we set  $v \in C_b^{2,2}(R^+ \times R)$  such that  $v \geq u$  and  $v(t, x) = u(t, x)$ . Notice that  $\hat{E}[a_\delta] = \bar{\mu}\delta$ ,  $-\hat{E}[-a_\delta] = \underline{\mu}\delta$ ,  $\hat{E}[a_\delta^2] = \bar{\mu}^2\delta^2$  and (2.7), by Taylor's expansion we have

$$\begin{aligned}
0 &\leq \hat{E}[v(t - \delta, x + a_\delta) - v(t, x)] \\
&= \hat{E}[-\partial_t v(t, x)\delta + \partial_x v(t, x)a_\delta + C(\delta^2 + a_\delta^2)] \\
&\leq -\partial_t v(t, x)\delta + \hat{E}[\partial_x v(t, x)a_\delta] + \hat{E}[C(\delta^2 + a_\delta^2)] \\
&\leq -\partial_t v(t, x)\delta + g(\partial_x v(t, x))\delta + C\delta^2,
\end{aligned}$$

from which we have  $\partial_t v(t, x) - g(\partial_x v(t, x)) \leq 0$ , we prove that  $u(t, x)$  is a viscosity subsolution of (2.9). We can prove that  $u(t, x)$  is also a viscosity supersolution of (2.9) in a similar way, thus  $u(t, x)$  is the viscosity solution of (2.9). It follows that  $a_t = \int_0^t \mu_t dt$  is  $N([\underline{\mu}t, \bar{\mu}t], \{0\})$  distributed. The proof is complete.  $\square$

Since the quadratic variation process  $\langle \hat{B} \rangle_t$  of the G-Brownian motion  $\hat{B}_t$  is  $N([\underline{\sigma}^2 t, \bar{\sigma}^2 t], \{0\})$  distributed, from Lemma 2.1 we have

**Lemma 2.2**  $\langle \hat{B} \rangle_t$  is identically distributed with  $\int_0^t \sigma_t dt$ , where for each fixed  $t \in R^+$   $\sigma_t$  is  $N([\underline{\sigma}^2, \bar{\sigma}^2], \{0\})$  distributed, and for  $t \geq 0$   $\sigma_t$  is independent from  $(\sigma_{t_1}, \sigma_{t_2}, \dots, \sigma_{t_n})$  for each  $n \in N$  and  $0 \leq t_1, \dots, t_n \leq t$ .

From Lemma 1 we have that  $b_t$  is identically distributed with  $\int_0^t \mu_s ds$ , then the price process of the stock can be rewritten as follows

$$dS_t = S_t(rdt + \theta_t dt + d\hat{B}_t), \quad (2.10)$$

where  $\theta_t = \mu_t - r$  is the risk premium which is the difference between the return rates of the stock and bond. It is easy to check that  $\theta_t$  is  $N([\underline{\mu} - r, \bar{\mu} - r], \{0\})$  distributed.

Consider an investor with wealth  $Y_t$  in the market, who can decide his invest portfolio and consumption at any time  $t \in [0, T]$ . We denote  $\pi_t$  as the amount of the wealth  $Y_t$  to invest in the stock at time  $t$ , and  $C(t+h) - C(t) \geq 0$  as the amount of money to withdraw for consumption during the interval  $(t, t+h]$ ,  $h > 0$ . We introduce the cumulative amount of consumption  $C_t$  as RCLL with  $C(0) = 0$ . We assume that all his decisions can only be based on the current path information  $\Omega_t$ .

**Definition 2.1** A self-financing superstrategy (resp. substrategy) is a vector process  $(Y, \pi, C)$  (resp.  $(-Y, \pi, C)$ ), where  $Y$  is the wealth process,  $\pi$  is the portfolio process, and  $C$  is the cumulative consumption process, such that

$$dY_t = rY_t dt + \pi_t d\hat{B}_t + \pi_t \theta_t dt - dC_t \quad (2.11)$$

$$(resp. -dY_t = -rY_t dt + \pi_t d\hat{B}_t + \pi_t \theta_t dt - dC_t) \quad (2.12)$$

where  $C$  is an increasing, right-continuous process with  $C_0 = 0$ . The superstrategy (resp. substrategy) is called feasible if the constraint of nonnegative wealth holds

$$Y_t \geq 0, \quad t \in [0, T].$$

### 3 Bid-ask pricing European contingent claim under uncertainty

From now on we consider a European contingent claim  $\xi$  written on the stock with maturity  $T$ , here  $\xi \in L_G^2(\Omega_T)$  is nonnegative. We give definitions of superhedging (resp. subhedging) strategy and ask (resp. bid) price of the claim  $\xi$ .

**Definition 3.1** (1) A superhedging (resp. subhedging) strategy against the European contingent claim  $\xi$  is a feasible self-financing superstrategy  $(Y, \pi, C)$  (resp. substrategy  $(-Y, \pi, C)$ ) such that  $Y_T = \xi$  (resp.  $-Y_T = -\xi$ ). We denote by  $\mathcal{H}(\xi)$  (resp.  $\mathcal{H}'(-\xi)$ ) the class of superhedging (resp. subhedging) strategies against  $\xi$ , and if  $\mathcal{H}(\xi)$  (resp.  $\mathcal{H}'(-\xi)$ ) is nonempty,  $\xi$  is called superhedgeable (resp. subhedgeable).

(2) The ask-price  $X(t)$  at time  $t$  of the superhedgeable claim  $\xi$  is defined as

$$X(t) = \inf\{x \geq 0 : \exists (Y_t, \pi_t, C_t) \in \mathcal{H}(\xi) \text{ such that } Y_t = x\},$$

and bid-price  $X'(t)$  at time  $t$  of the subhedgeable claim  $\xi$  is defined as

$$X'(t) = \sup\{x \geq 0 : \exists (-Y_t, \pi_t, C_t) \in \mathcal{H}'(-\xi) \text{ such that } -Y_t = -x\}.$$

Under uncertainty, the market is incomplete and the superhedging (resp. subhedging) strategy of the claim is not unique. The definition of the ask-price  $X(t)$  implies that the ask-price  $X(t)$  is the minimum amount of risk for the buyer to superhedging the claim, then it is coherent measure of risk of all superstrategies against the claim for the buyer. The coherent risk measure of all superstrategies against the claim can be regard as the sublinear expectation of the claim, we have the following representation of bid-ask price of the claim.

**Theorem 3.1** *Let  $\xi \in L_G^2(\Omega_T)$  be a nonnegative European contingent claim. There exists a superhedging (resp. subhedging) strategy  $(X, \pi, C) \in \mathcal{H}(\xi)$  (resp.  $(-X', \pi, C) \in \mathcal{H}'(-\xi)$ ) against  $\xi$  such that  $X_t$  (resp.  $X'_t$ ) is the ask (resp. bid) price of the claim at time  $t$ .*

*Let  $(H_s^t : s \geq t)$  be the deflator started at time  $t$  and satisfy*

$$dH_s^t = -H_s^t[rds + \frac{\theta_s}{\sigma_s}d\hat{B}_s], \quad H_t^t = 1. \quad (3.13)$$

*Then the ask-price against  $\xi$  at time  $t$  is*

$$X_t = \hat{E}[H_T^t \xi | \Omega_t],$$

*and the bid-price against  $\xi$  at time  $t$  is*

$$X'_t = -\hat{E}[-H_T^t \xi | \Omega_t],$$

**Proof.** By G-Itô's formula we can check that

$$H_t = \exp\{-[\int_0^t rds + \int_0^t \frac{\theta_s}{\sigma_s}d\hat{B}_s + \frac{1}{2} \int_0^t (\frac{\theta_s}{\sigma_s})^2 d <\hat{B}>_s]\} \quad (3.14)$$

is the solution of (3.13). Define the stochastic process  $X$  from

$$H_t X_t = \hat{E}[H_T \xi | \Omega_t] = M_t,$$

then  $M_t$  is a G martingale. By the G-martingale representation theorem ([25]), there exists unique decomposition of  $H_t X_t$  as follows

$$H_t X_t = \hat{E}[H_T \xi] + \int_0^t \beta_s d\hat{B}_s - K_t,$$

where  $\{\beta_t\} \in H_G^1(0, T)$ ,  $\{K_t\}$  is a continuous, increasing process with  $K_0 = 0$ , and  $\{-K_t\}_{0 \leq t \leq T}$  is a G-martingale. Set  $\pi_t = [H_t^{-1} \beta_t + X_t \frac{\theta_t}{\sigma_t}]$ . Then  $H_t X_t = \hat{E}[H_T \xi] + \int_0^t H_s (\pi_s - X_s \frac{\theta_s}{\sigma_s}) d\hat{B}_s - K_t$ . Define  $C(t) = \int_0^t H_s^{-1} K_s ds$ , then  $C(t)$  is nonnegative and increasing process with  $C(0) = 0$ . We prove that  $(\hat{E}[H_T^t \xi | \Omega_t], \pi_t, C_t) \in \mathcal{H}(\xi)$  is a superhedging strategy against  $\xi$ .

For any superhedging strategy  $(Y_t, \hat{\pi}_t, \hat{C}_t)$  against  $\xi$ , by G-Itô's formula and  $\sigma_t dt = d <\hat{B}>_t$  which given by Lemma 2.2, we have

$$H_t Y_t = H_T \xi - \int_t^T H_s (\hat{\pi}_s - Y_s \frac{\theta_s}{\sigma_s}) d\hat{B}_s + \int_t^T H_s d\hat{C}_s,$$

Taking the condition G-expectation on both side of (3.15) with respect to  $\Omega_t$ , notice that  $\hat{C}_t$  is a nonnegative right continuous process and  $\hat{E}[\int_t^T H_s (\pi_s - Y_s \frac{\theta_s}{\sigma_s}) d\hat{B}_s | \Omega_t] = 0$ , we have

$$H_t Y(t) \geq \hat{E}[H_T \xi | \Omega_t],$$

i.e.

$$Y(t) \geq \hat{E}[H_T^t \xi | \Omega_t] = X_t,$$

which prove that  $X_t = \hat{E}[H_T^t \xi | \Omega_t]$  is the ask price against the claim  $\xi$  at time  $t$ . Similarly we can prove that  $X'_t = -\hat{E}[-H_T^t \xi | \Omega_t]$  is the bid price against the claim  $\xi$  at time  $t$ .  $\square$

## 4 G martingale transform and bid-ask dynamic pricing mechanisms

In this section, we will construct a time consistent G expectation  $E^G$ , and transfer the uncertainty model into the sublinear space  $(\Omega, L_G^2(\Omega), E^G)$  which corresponds with a sequence of the risk-neutral probability measure space.

Define a sublinear function  $G(\cdot)$  as follows

$$G(\alpha) = \frac{1}{2}(\bar{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-), \quad \forall \alpha \in R. \quad (4.15)$$

For given  $\varphi \in C_{b, \text{lip}}(R)$ , we denote  $u(t, x)$  as the viscosity solution of the following G-heat equation

$$\begin{aligned} \partial_t u - G(\partial_{xx} u) &= 0, & (t, x) &\in (0, \infty) \times R \\ u(0, x) &= \varphi(x). \end{aligned} \quad (4.16)$$

**Remark 2** The G-heat equation (4.16) is a special kind of Hamilton-Jacobi-Bellman equation, also the Barenblatt equation except the case  $\underline{\sigma} = 0$  (see [3] and [4]). The existence and uniqueness of (4.16) in the sense of viscosity solution can be found in, for example [13], [10], and [17] for  $C^{1,2}$ -solution if  $\underline{\sigma} > 0$ .

Denote

$$\tilde{B}_t = B_t - rt, \quad (4.17)$$

the stochastic path information of  $\{\tilde{B}_t\}_{t \geq 0}$  up to  $t$  is the same as  $\{B_t\}_{t \geq 0}$ , without loss of generality we still denote  $\Omega_t$  as the path information of  $\{\tilde{B}_t\}_{t \geq 0}$  up to  $t$ . For  $\omega \in \Omega$  consider the process  $(\tilde{B}_t)(\omega) = \omega_t, t \in [0, \infty)$ , we define  $E^G[\cdot] : L_{ip}(\Omega) \longrightarrow R$  as

$$E^G[\varphi(\tilde{B}_t)] := u(t, 0),$$

and for each  $s, t \geq 0$  and  $t_1, \dots, t_N \in [0, t]$

$$E^G[\varphi(\tilde{B}_{t_1}, \dots, \tilde{B}_{t_N}, \tilde{B}_{t+s} - \tilde{B}_t)] := E^G[\psi(\tilde{B}_{t_1}, \dots, \tilde{B}_{t_N})]$$

where  $\psi(x_1, \dots, x_N) = E^G[\varphi(x_1, \dots, x_N, \tilde{B}_s)]$ .

For  $0 < t_1 < t_2 < \dots < t_i < t_{i+1} < \dots < t_N < +\infty$ , we define condition G expectation with respect to  $\Omega_{t_i}$  as

$$\begin{aligned} &E^G[\varphi(\tilde{B}_{t_1}, \tilde{B}_{t_2} - \tilde{B}_{t_1}, \dots, \tilde{B}_{t_{i+1}} - \tilde{B}_{t_i}, \dots, \tilde{B}_{t_N} - \tilde{B}_{t_{N-1}}) | \Omega_{t_i}] \\ &:= \psi(\tilde{B}_{t_1}, \tilde{B}_{t_2} - \tilde{B}_{t_1}, \dots, \tilde{B}_{t_i} - \tilde{B}_{t_{i-1}}), \end{aligned}$$

where  $\psi(x_1, \dots, x_i) = E^G[\varphi(x_1, \dots, x_i, \tilde{B}_{t_{i+1}} - \tilde{B}_{t_i}, \dots, \tilde{B}_{t_N} - \tilde{B}_{t_{N-1}})]$ .

We consistently define a sublinear expectation  $E^G$  on  $L_{ip}(\Omega)$ . Under the sublinear expectation  $E^G$  we define above, the corresponding process  $(\tilde{B}_t)_{t \geq 0}$  is a G-Brownian motion and  $\tilde{B}_t$  is  $N(\{0\}, [\underline{\sigma}^2 t, \bar{\sigma}^2 t])$  distributed. We call  $E^G$  as G-expectation on  $L_{ip}(\Omega)$ .

Denote  $L_G^p(\Omega), p \geq 1$  as the completion of  $L_{ip}(\Omega)$  under the norm  $\|X\|_p = (E^G[|X|^p])^{1/p}$ , and similarly we can define  $L_G^p(\Omega_t)$ .  $E^G$  can be continuously extended to the space  $(\Omega, L_G^1(\Omega))$ . From now on we will work in the sublinear expectation space  $(\Omega, L_G^1(\Omega), E^G)$ .



The price dynamic process of the stock (2.4) can be rewritten as follows

$$dS_t = S_t(rdt + d\tilde{B}_t). \quad (4.18)$$

Denote  $D(t) := e^{-rt}$  be the discounted factor, with the discounted processes  $\bar{Y}_t = D(t)Y_t$ ,  $\bar{\pi}_t = D(t)\pi_t$ , and  $d\bar{C}_t = D(t)dC_t$ , using G-Itô's formula, we can write the self-financing superstrategy  $(\bar{Y}, \bar{\pi}, \bar{C})$  (resp. substrategy  $(-\bar{Y}, \bar{\pi}, \bar{C})$ ) satisfying

$$d\bar{Y}_t = \bar{\pi}_t d\bar{B}_t - d\bar{C}_t.$$

$$(\text{resp. } -d\bar{Y}_t = \bar{\pi}_t d\bar{B}_t - d\bar{C}_t).$$

The superhedging (resp. subhedging) strategies and ask (resp. bid) price of the claim  $\xi$  which we defined in Definition 3.1 can also be characterized using discounted quantities.

For the nonnegative European contingent claim  $\xi \in L_G^2(\Omega_T)$ , we define G dynamic pricing mechanism for the claim  $\xi$  as follows

**Definition 4.1** For  $t \in [0, T]$ , we define G dynamic pricing mechanism as  $E_{t,T}^G : L_G^2(\Omega_T) \longrightarrow L_G^2(\Omega_t)$

$$E_{t,T}^G[\cdot] = E^G[\cdot | \mathcal{F}_t].$$

By the comparison theorem of the G-heat equation (4.16) and the sublinear property of the function  $G(\cdot)$ , the G dynamic pricing mechanisms have the following properties

**Proposition 4.1** For  $t \in [0, T]$  and  $\xi_1, \xi_2 \in L_G^2(\Omega_T)$

- (i)  $E_{t,T}^G[\xi_1] \geq E_{t,T}^G[\xi_2]$ , if  $\xi_1 \geq \xi_2$ ;
- (ii)  $E_{T,T}^G[\xi] = \xi$ ;
- (iii)  $E_{t,T}^G[\xi_1 + \xi_2] \leq E_{t,T}^G[\xi_1] + E_{t,T}^G[\xi_2]$ ;
- (iv)  $E_{t,T}^G[\lambda \xi] = \lambda E_{t,T}^G[\xi]$  for  $\lambda \geq 0$ ;
- (v)  $E_{s,t}^G[E_{t,T}^G[\xi]] = E_{s,T}^G[\xi]$  for  $0 \leq s \leq t$ .

**Theorem 4.1** Assume that  $\xi = \phi(S_T) \in L_G^2(\Omega_T)$  be a nonnegative European contingent claim, and  $E_{t,T}^G[\cdot]$  be the G pricing dynamic mechanisms defined in Definition 4.1. The ask price and bid price against the contingent claim  $\xi$  at time  $t$  are

$$u^a(t, S_t) = e^{-r(T-t)} E_{t,T}^G[\xi] \quad \text{and} \quad u^b(t, S_t) = -e^{-r(T-t)} E_{t,T}^G[-\xi] \quad (4.19)$$

respectively.

**Proof.** It is easy to check that  $\{u_t^a\}_{0 \leq t \leq T}$  satisfying

$$D(t)u_t^a = E_{t,T}^G[D(T)\xi],$$

then  $M_t = D(t)u_t^a$  is a G-martingale. By a similar way we used in the proof of Theorem 3.1, we can complete the proof.  $\square$

**Remark 3** In the Properties 4.1, (iii) and (iv) imply that

$$E_{t,T}^G[\alpha \xi_1 + (1 - \alpha) \xi_2] \leq \alpha E_{t,T}^G[\xi_1] + (1 - \alpha) E_{t,T}^G[\xi_2], \text{ for } \alpha \in [0, 1],$$

which means that the G dynamic pricing mechanism is a convex pricing mechanism. (v) means the G dynamic pricing mechanism is a time consistent pricing mechanism, under the G dynamic pricing mechanism the ask price  $u^a(s, S_s) = e^{-r(T-s)} E_{s,T}^G[\xi]$  (resp. the bid price  $u^b(s, S_s) = -e^{-r(T-s)} E_{s,T}^G[-\xi]$ ) at time  $s$  ( $s \leq t \leq T$ ) against the claim  $\xi$  with maturity  $T$  could be regards as the ask (resp. bid) price at time  $s$  against the claim  $u^a(t, S_t) = e^{-r(T-t)} E_{t,T}^G[\xi]$  (resp.  $u^b(t, S_t) = -e^{-r(T-t)} E_{t,T}^G[-\xi]$ ) with maturity  $t$ .

## 5 Markovian case

We assume that the stock price dynamic satisfying the following SDE ( $t \geq 0$ )

$$\begin{aligned} dS_s^{t,x} &= S_s^{t,x}(rdt + d\tilde{B}_t), \quad s \in [t, T], \\ S_t^{t,x} &= x. \end{aligned} \quad (5.20)$$

For given a nonnegative European contingent claim  $\xi = \phi(S_T) \in L_G^2(\Omega)$ , from Theorem 4.1 its ask price and bid price at time  $t$  are

$$\begin{aligned} u^a(t, x) &= e^{-r(T-t)} E_{t,T}^G[\phi(S_T^{t,x})], \\ u^b(t, x) &= -e^{-r(T-t)} E_{t,T}^G[-\phi(S_T^{t,x})], \end{aligned}$$

we establish the relation between ask (resp. bid) price with the viscosity solution of a full nonlinear PDE.

**Theorem 5.1** *Assume that the stock price dynamic process satisfying (5.20),  $\xi = \phi(S_T^{t,x}) \in L_G^2(\Omega_T)$  be a nonnegative European contingent claim written on the stock with maturity  $T$ , and  $\phi : R \rightarrow R$  be a given Lipschitz function. The ask price of the contingent claim  $u^a(t, x) = e^{-r(T-t)} E_{t,T}^G[\xi]$  is the viscosity solution of the following nonlinear PDE*

$$\begin{aligned} \partial_t u^a(t, x) + rx \partial_x u^a(t, x) + G(x^2 \partial_{xx} u^a(t, x)) - ru^a(t, x) &= 0, \quad (t, x) \in [0, T) \times R \\ u^a(T, x) &= \phi(x). \end{aligned} \quad (5.21)$$

*The bid price of the contingent claim  $u^b(t, x) = -e^{-r(T-t)} E_{t,T}^G[-\xi]$  is the viscosity solution of the following nonlinear PDE*

$$\begin{aligned} \partial_t u^b(t, x) + rx \partial_x u^b(t, x) - G(-x^2 \partial_{xx} u^b(t, x)) - ru^b(t, x) &= 0, \quad (t, x) \in [0, T) \times R \\ u^b(T, x) &= \phi(x). \end{aligned} \quad (5.22)$$

**Proof.** With the assumption that  $\xi \in L_G^2(\Omega_T)$ , Peng in [23] prove that  $v(t, x) = E_{t,T}^G[\phi(S_T^{t,x})]$  satisfying

$$|v(t, x) - v(t, x')| \leq C|x - x'|, \quad |v(t, x)| \leq C(1 + |x|),$$

and for  $\delta \in [0, T - t]$

$$|v(t, x) - v(t + \delta, x)| \leq C(1 + |x|)(\delta^{1/2} + \delta), \quad \delta \in [0, T - t],$$

$$v(t, x) = E^G[v(t + \delta, S_{t+\delta}^{t,x})],$$

where  $C$  is only dependent on the Lipschitz constant.

We can easily get that

$$\begin{aligned} |u^a(t, x) - u^a(t, x')| &\leq C|x - x'|, \quad |u^a(t, x)| \leq C(1 + |x|), \\ |u^a(t, x) - u^a(t + \delta, x)| &\leq C(1 + |x|)(\delta^{1/2} + \delta), \end{aligned}$$

and

$$u^a(t, x) = E^G[e^{-r\delta} u^a(t + \delta, S_{t+\delta}^{t,x})].$$

For fixed  $(t, x) \in (0, T) \times R$ , let  $\psi \in C_b^{2,3}([0, T] \times R)$  be such that  $\psi \geq u^a$  and  $\psi(t, x) = u^a(t, x)$ . By Taylor's expansion, we have for  $\delta \in (0, T - t)$

$$\begin{aligned}
0 &\leq E^G[e^{-r\delta}\psi(t + \delta, S_{t+\delta}^{t,x}) - \psi(t, x)] \\
&\leq \frac{1}{2}E^G[x^2\partial_{xx}\psi(t, x)(\langle B \rangle_{t+\delta} - \langle B \rangle_t)] \\
&\quad + (\partial_t\psi(t, x) + rx\partial_x\psi(t, x) - r\psi(t, x))\delta + C(1 + |x| + |x|^2 + |x|^3)\delta^{3/2} \\
&\leq (\partial_t\psi(t, x) + rx\partial_x\psi(t, x) + G(x^2\partial_{xx}\psi(t, x)) - r\psi(t, x))\delta \\
&\quad + C(1 + |x| + |x|^2 + |x|^3)\delta^{3/2},
\end{aligned}$$

for  $\delta \downarrow 0$ , we have

$$\partial_t\psi(t, x) + rx\partial_x\psi(t, x) + G(x^2\partial_{xx}\psi(t, x)) - r\psi(t, x) \geq 0,$$

which implies that  $u^a(t, x)$  is the subsolution of the nonlinear PDE (5.21), and by a similar way we can prove that  $u^a(t, x)$  is the supersolution of (5.21). Thus, we prove that the ask price against the claim  $\xi$  at time  $t$  is the viscosity solution of (5.21). Similarly, we can prove that  $u^b(t, x) = -e^{-r(T-t)}E_{t,T}^G[-\phi(S_T^{t,x})]$  is the viscosity solution of (5.22).  $\square$

Assume that the stoke price solve the following SDE

$$\begin{aligned}
d \log \tilde{S}_u^{t,x} &= rdt + \tilde{\sigma}_t dW_t, u \in [t, T], \\
\tilde{S}_t^{t,x} &= x,
\end{aligned} \tag{5.23}$$

where  $(W_t)_{t \geq 0}$  be a 1-dimensional standard Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, P)$  and the filtration generated by  $W_t$  is  $(\mathcal{F}_t : 0 \leq t \leq T)$ . Assume that  $(\tilde{\sigma}_t)_{t \geq 0}$  is an adapted process such that  $\tilde{\sigma}_t \in [\underline{\sigma}, \bar{\sigma}]$ . It is well known that the Black-Scholes type price against the European contingent claim  $\xi = \phi(\tilde{S}_T^{t,x})$  at time  $t$  is  $u(t, x) = e^{-r(T-t)}E[\phi(\tilde{S}_T^{t,x})|\mathcal{F}_t]$  which is the solution of the following PDE

$$\begin{aligned}
\partial_t u(t, x) + rx\partial_x u(t, x) + \frac{1}{2}x^2\tilde{\sigma}_t^2\partial_{xx}u(t, x) - ru(t, x) &= 0, \quad (t, x) \in R^+ \times R \\
u(T, x) &= \phi(x).
\end{aligned} \tag{5.24}$$

**Corollary 5.1** Assume that  $\phi : R \rightarrow R$  be a given Lipschitz function,  $u(t, x)$  is the Black-Scholes type price against the claim  $\phi(\tilde{S}_T^{t,x})$  which satisfies ??black). Assume that  $u^b(t, x)$  and  $u^a(t, x)$  satisfying (5.22) and (5.21), i.e., be the bid price and ask price against the European contingent claim  $\xi = \phi(\tilde{S}_T^{t,x})$  at time  $t$ . Then we have

$$u^b(t, x) \leq u(t, x) \leq u^a(t, x). \tag{5.25}$$

**Proof.** By using the comparison theorem proposed by Peng in [23], we can easily prove the Lemma.  $\square$

**Lemma 5.1** Assume that the price process  $(S_s^{t,x})_{s \geq t}$  of the stock satisfies (5.20),  $\xi = \phi(S_T^{t,x}) \in L_G^2(\Omega_T)$  be a nonnegative European contingent claim written on stock with maturity  $T$ , and  $\phi : R \rightarrow R$  be a given Lipschitz function.

(I) If  $\phi(\cdot)$  is convex (resp. concave), for any  $t \in [0, T]$  the ask price function  $u^a(t, \cdot)$  is convex (resp. is concave), and  $u^a(t, x)$  satisfies (5.21) with  $G(x^2\partial_{xx}u^a) = \frac{1}{2}\bar{\sigma}^2x^2\partial_{xx}u^a$  (resp.  $= \frac{1}{2}\underline{\sigma}^2x^2\partial_{xx}u^a$ ). If  $\phi(\cdot)$  is convex (resp. concave) on interval  $(a, b) \in R$  ( $a < b$ ) the ask price function  $u^a(t, \cdot)$  is convex (resp. concave) on  $(a, b)$  for any  $t \in [0, T]$ .

(II) If  $\phi(\cdot)$  is convex (resp. concave), for any  $t \in [0, T]$  the bid price function  $u^b(t, \cdot)$  is convex (resp. is concave), and  $u^b(t, x)$  satisfies (5.22) with  $G(-x^2 \partial_{xx} u^b) = -\frac{1}{2} \underline{\sigma}^2 x^2 \partial_{xx} u^b$  (resp.  $= -\frac{1}{2} \bar{\sigma}^2 x^2 \partial_{xx} u^b$ ). If  $\phi(\cdot)$  is convex (resp. concave) on interval  $(a, b) \in R$  ( $a < b$ ) the bid price function  $u^b(t, \cdot)$  is convex (resp. concave) on  $(a, b)$  for any  $t \in [0, T]$ .

**Proof.** We only prove (I), the proof of (II) is similar with the proof of (I).

1. First we prove that  $u^a(t, x)$  is convex if  $\phi(x)$  is convex.

For  $x_1, x_2 \in R$  and for given  $\alpha \in [0, 1]$ , by G-Itô's formula, we get the stock price process as follows

$$\begin{aligned}
& S_T^{t, \alpha x_1 + (1-\alpha)x_2} \\
&= (\alpha x_1 + (1-\alpha)x_2) \exp(r(T-t) + \tilde{B}_T - \tilde{B}_t - \frac{1}{2}(\langle \tilde{B} \rangle_T - \langle \tilde{B} \rangle_t)) \\
&= \alpha x_1 \exp(r(T-t) + \tilde{B}_T - \tilde{B}_t - \frac{1}{2}(\langle \tilde{B} \rangle_T - \langle \tilde{B} \rangle_t)) \\
&\quad + (1-\alpha)x_2 \exp(r(T-t) + \tilde{B}_T - \tilde{B}_t - \frac{1}{2}(\langle \tilde{B} \rangle_T - \langle \tilde{B} \rangle_t)) \\
&= \alpha S_T^{t, x_1} + (1-\alpha) S_T^{t, x_2}.
\end{aligned} \tag{5.26}$$

Since the G pricing dynamic mechanism is a convex pricing mechanism, we have that

$$\begin{aligned}
& u^a(t, \alpha x_1 + (1-\alpha)x_2) \\
&= e^{-r(T-t)} E_{t,T}^G[\phi(S_T^{t, \alpha x_1 + (1-\alpha)x_2})] \\
&= e^{-r(T-t)} E_{t,T}^G[\phi(\alpha S_T^{t, x_1} + (1-\alpha) S_T^{t, x_2})] \\
&\leq e^{-r(T-t)} E_{t,T}^G[\alpha \phi(S_T^{t, x_1}) + (1-\alpha) \phi(S_T^{t, x_2})] \\
&\leq \alpha e^{-r(T-t)} E_{t,T}^G[\phi(S_T^{t, x_1})] + (1-\alpha) e^{-r(T-t)} E_{t,T}^G[\phi(S_T^{t, x_2})] \\
&= \alpha u^a(t, x_1) + (1-\alpha) u^a(t, x_2),
\end{aligned} \tag{5.27}$$

which prove that  $u^a(t, x)$  is convex on  $R$ ,  $\partial_{xx} u^a$  is nonnegative on  $R$  and  $G(x^2 \partial_{xx} u^a) = \frac{1}{2} \bar{\sigma}^2 x^2 \partial_{xx} u^a$ .

If  $\phi(x)$  is convex in interval  $(a, b)$ , for  $x_1, x_2 \in (a, b)$ , (5.26), (5.27) hold on  $(a, b)$  and  $u^a(t, x)$  is convex on  $(a, b)$ .

2. We will prove that  $u^a(t, x)$  is concave if  $\phi(x)$  is concave.

For given process  $(\sigma_t)_{t \geq 0}$  such that  $\sigma_t \in [\underline{\sigma}, \bar{\sigma}]$ , we denote  $u_\sigma^a(t, x)$  the solution of the following PDE

$$\begin{aligned}
\partial_t u_\sigma^a(t, x) + r x \partial_x u_\sigma^a(t, x) + \frac{1}{2} \sigma_t^2 x^2 \partial_{xx} u_\sigma^a(t, x) &= 0, \quad (t, x) \in R^+ \times R \\
u_\sigma^a(T, x) &= \phi(x),
\end{aligned}$$

then  $u_\sigma^a(t, x) = E_t[\phi(x \exp[r(T-t) + \sigma_t(W_T - W_t) - \frac{1}{2} \sigma_t^2(T-t)])]$ , where  $W_t$  is the standard Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), \bar{P})$ , and  $E_t$  is the corresponding condition expectation. Denote  $S_{T,\sigma}^{t,x} = x \exp[r(T-t) + \sigma_t(W_T - W_t) - \frac{1}{2} \sigma_t^2(T-t)]$ , for  $x_1, x_2 \in R$  if  $\phi(x)$  is concave

on  $R$ , we have

$$\begin{aligned}
& u_{\sigma}^a(t, \alpha x_1 + (1 - \alpha)x_2) \\
&= E_t[\phi((\alpha x_1 + (1 - \alpha)x_2) \exp[r(T - t) + \sigma_t(W_T - W_t) - \frac{1}{2}\sigma_t^2(T - t)])] \\
&= E_t[\phi(\alpha S_{T,\sigma}^{t,x_1} + (1 - \alpha)S_{T,\sigma}^{t,x_2})] \\
&\geq E_t[\alpha \phi(S_{T,\sigma}^{t,x_1}) + (1 - \alpha)\phi(S_{T,\sigma}^{t,x_2})] \\
&= \alpha E_t[\phi(S_{T,\sigma}^{t,x_1})] + (1 - \alpha)E_t[\phi(S_{T,\sigma}^{t,x_2})] \\
&= \alpha u_{\sigma}^a(t, x_1) + (1 - \alpha)u_{\sigma}^a(t, x_2),
\end{aligned} \tag{5.28}$$

which mean  $u_{\sigma}^a(t, \cdot)$  is concave on  $R$ , and the function  $U(t, x) = \sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} u_{\sigma}^a(t, x)$  is concave on  $R$ , i.e.

$$U(t, \alpha x_1 + (1 - \alpha)x_2) \geq \alpha U(t, x_1) + (1 - \alpha)U(t, x_2). \tag{5.29}$$

Since the operator  $\sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} E_t$  is a convex operator and  $U(t, x) = \sup_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} E_t[\phi(S_{T,\sigma}^{t,x})]$ , using the similar argument in Theorem 5.1, we can prove that  $U(t, x)$  is the viscosity solution of the following HJB equation

$$\begin{aligned}
\partial_t U(t, x) + rx\partial_x U(t, x) + \frac{1}{2} \sup_{\sigma_t \in [\underline{\sigma}, \bar{\sigma}]} [\sigma_t^2 x^2 \partial_{xx} U(t, x)] &= 0, \quad (t, x) \in R^+ \times R \\
U(T, x) &= \phi(x),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\partial_t U(t, x) + rx\partial_x U(t, x) + G(x^2 \partial_{xx} U(t, x)) &= 0, \quad (t, x) \in R^+ \times R \\
U(T, x) &= \phi(x).
\end{aligned} \tag{5.30}$$

In [23], Peng prove that  $U(t, x) = E_{t,T}^G[\phi(S_T^{t,x})]$  is the unique viscosity solution of (5.30), from (5.29) the solution  $U(t, x)$  is concave for  $x$  on  $R$ . Thus we prove that  $u^a(t, x) = e^{-r(T-t)}U(t, x)$  is concave for  $x$  on  $R$  and  $G(x^2 \partial_{xx} u^a) = \frac{1}{2}\sigma^2 x^2 \partial_{xx} u^a$ .

If  $\phi(x)$  is concave on  $(a, b)$ , (5.28) and (5.29) hold on  $(a, b)$ , which prove that  $u^a(t, x)$  is concave on  $(a, b)$ . We finish the proof of (I).  $\square$

## 6 Monotone characteristic finite difference schemes

In this section we will consider numerical schemes for the nonlinear PDE (5.21) (similar for (5.22)).

### 6.1 Characteristic finite difference schemes

Define  $u(\tau, x) = u^a(T - t, x)$ , the nonlinear PDE (5.21) (similar for (5.22)) can be written as

$$\begin{aligned}
\partial_{\tau} u(\tau, x) - rx\partial_x u(\tau, x) - x^2 G(\partial_{xx} u(\tau, x)) + ru(\tau, x) &= 0, \quad (\tau, x) \in [0, T] \times R, \\
u(0, x) &= \phi(x).
\end{aligned} \tag{6.1}$$

First, we consider the boundary conditions of (6.1). As  $x \rightarrow 0$ , equation (6.1) becomes

$$\partial_{\tau} u|_{x=0} = -ru(\tau, 0). \tag{6.2}$$

For  $x \rightarrow \infty$ , normally for sufficient big enough  $x$  there holds  $\partial_{xx}u \simeq 0$ . We set  $x = S_{\max}$  with  $S_{\max}$  big enough such that the payoff of the contingent claim has asymptotic form. We consider the Dirichlet condition as follows

$$u|_{x=S_{\max}} = g(\tau, S_{\max}), \quad (6.3)$$

where  $g(., l)$  can be determined by financial reasoning for some given contingent claim, normally in the following asymptotic form

$$g(\tau, S_{\max}) = b(\tau)S_{\max} + c(\tau).$$

We assume that  $b(\tau), c(\tau)$  are bounded such that

$$|u(\tau, S_{\max})| \leq C_b, \quad (6.4)$$

where  $C_b$  is a constant.

There is convection term  $\partial_\tau u(\tau, x) - rx\partial_x u(\tau, x)$  in (6.1), if the convection dominate the diffusion finite difference discretization for (6.1) could leads numerical oscillation, we consider discrete the convection term along the characteristic direction ([9]). Denote  $\psi(x) = [1 + r^2 x^2]^{1/2}$ , the direction derivative along characteristic direction is  $\frac{\partial}{\partial c} = \frac{1}{\psi}(\frac{\partial}{\partial \tau} - rx\frac{\partial}{\partial x})$ . (6.1)-(6.3) is equivalent to the following PDE

$$\begin{cases} \psi(x)\frac{\partial u}{\partial c} - \sup_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \mathcal{L}^\sigma u = 0, & (\tau, x) \in (0, T] \times (0, S_{\max}), \\ \partial_\tau u|_{x=0} = -ru(\tau, 0), & u|_{x=S_{\max}} = g(\tau, S_{\max}), \\ u(0, x) = \phi(x), \end{cases} \quad (6.5)$$

where  $\mathcal{L}^\sigma u = \frac{\sigma^2}{2}x^2\partial_{xx}u - ru$ .

Now we define space partition of  $I = [0, S_{\max}]$ . Let  $I = [0, S_{\max}]$  be divided into  $N$  sub-intervals

$$I_i = (x_i, x_{i+1}), \quad i = 0, \dots, N-1,$$

with  $0 = x_0 < x_1 < \dots < x_N = S_{\max}$ . For each  $i = 0, \dots, N-1$ , let  $\Delta x_i = x_{i+1} - x_i$ . Let  $\{t_i\}_{i=0}^M$  be a set of partition point in  $[0, T]$  satisfying  $0 = t_0 < t_1 < \dots < t_M = T$ , and denote  $\Delta t_n = t_n - t_{n-1} > 0$ , where  $M > 1$  is a positive integer.

Let  $u_i^n$  be a discrete approximation to  $u(t_n, x_i)$ . Denote  $\bar{x}_i^n = x_i + rx_i\Delta t_{n+1}$ , for  $\Delta t_{n+1}$  small enough such that  $\bar{x}_i^n \in [x_i, x_{i+1}]$ . Denote  $\bar{u}_i^n$  be a discrete approximation to  $u(t_n, \bar{x}_i^n)$ , here we define  $\bar{u}_i^n$  as the linear interpolate function of  $u_i^n$ . The implicit characteristic finite difference scheme for (6.1) is as follows:

For boundary  $x = 0$

$$\frac{u_0^{n+1} - u_0^n}{\Delta t_{n+1}} = -ru_0^{n+1}, \quad (6.6)$$

for  $i = 1, 2, \dots, N-1$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t_{n+1}} = \sup_{\sigma^{n+1} \in \{\underline{\sigma}, \bar{\sigma}\}} [(L_h^{\sigma^{n+1}} u^{n+1})_i] + \frac{\bar{u}_i^n - u_i^n}{\Delta t_{n+1}}, \quad (6.7)$$

where  $(L_h^{\sigma^n} u^n)_i$  denotes the central difference discrete of  $(L^\sigma u)$  at note  $(t_n, x_i)$ , i.e.

$$(L_h^{\sigma^n} u^n)_i = \alpha_i^n(\sigma_i^n)u_{i-1}^n + \beta_i^n(\sigma_i^n)u_{i+1}^n - (\alpha_i^n(\sigma_i^n) + \beta_i^n(\sigma_i^n) + r)u_i^n, \quad (6.8)$$

where  $\alpha_i^n$  and  $\beta_i^n$  are defined as follows

$$\begin{aligned}\alpha_i^n(\sigma_i^n) &= \frac{(\sigma_i^n)^2 x_i^2}{(x_i - x_{i-1})(x_{i+1} - x_{i-1})}, \\ \beta_i^n(\sigma_i^n) &= \frac{(\sigma_i^n)^2 x_i^2}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}.\end{aligned}\tag{6.9}$$

It is easy to check that  $\alpha_i^n, \beta_i^n \geq 0$ . We define

$$\begin{aligned}U^n &= [u_0^n, u_1^n, \dots, u_{N-1}^n, u_N^n]^T, \\ \bar{U}^n &= [\bar{u}_0^n, \bar{u}_1^n, \dots, \bar{u}_{N-1}^n, \bar{u}_N^n]^T, \\ \sigma^n &= [\sigma_1^n, \dots, \sigma_{N-1}^n]^T,\end{aligned}$$

and

$$(A^n(\sigma^n)U^n)_i = \alpha_i^n(\sigma_i^n)u_{i-1}^n + \beta_i^n(\sigma_i^n)u_{i+1}^n - (\alpha_i^n(\sigma_i^n) + \beta_i^n(\sigma_i^n) + r)u_i^n.$$

For notational consistency, we denote  $\bar{u}_0^n = \frac{1}{1+r\Delta t_{n+1}}u_0^n$ ,  $\bar{u}_N^n = u_N^{n+1}$ , and enforce the first row and the last row of  $A$  to be zero. Then the discretization scheme (6.7) can be write as the following equivalent matrix form

$$\begin{cases} [I - \Delta t_{n+1}\hat{A}^{n+1}]U^{n+1} = \bar{U}^n \\ \hat{\sigma}_i^{n+1} = \operatorname{argsup}_{\sigma^{n+1} \in \{\underline{\sigma}, \bar{\sigma}\}} \{(A^{n+1}U^{n+1})_i\} \end{cases}\tag{6.10}$$

where  $\hat{A}^{n+1} = A(\hat{\sigma}^{n+1})$  and  $\hat{\sigma}^{n+1} = \hat{\sigma}^{n+1}(U^{n+1})$ .

It is easy to see that  $-\hat{A}^{n+1}$  has nonpositive off-diagonals, positive diagonal, and is diagonally dominant. We have the following theorem

**Theorem 6.1** *Matrices  $-\hat{A}^{n+1}$  and  $I - \Delta t^{n+1}\hat{A}^{n+1}$  are M-matrices ([26]).*

The equation (6.1) has unique viscosity solution, and satisfies the strong comparison property (see [6] and [10]), then a numerical scheme converges to the viscosity solution if the method is consistent, stable and monotone.

Let  $h = \max\{\Delta x, \Delta t\}$  be the mesh parameter, where  $\Delta x = \max_i \Delta x_i$ ,  $\Delta t = \max_n \Delta t_n$ . Assume that the partition is quasi-uniform, i.e.,  $\exists C_1, C_2 > 0$  independent of  $h$ , such that

$$C_1 h \leq \Delta x_i, \Delta t_n \leq C_2 h\tag{6.11}$$

for  $0 \leq i \leq N-1$  and  $1 \leq n \leq M$ .

We denote

$$G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) = \frac{u_i^{n+1} - u_i^n}{\Delta t_{n+1}} - \sup_{\sigma^{n+1} \in \{\underline{\sigma}, \bar{\sigma}\}} (A^{n+1}U^{n+1})_i - \frac{\bar{u}_i^n - u_i^n}{\Delta t_{n+1}},\tag{6.12}$$

then the discrete equation at each node can be written as the following form

$$G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) = 0\tag{6.13}$$

**Lemma 6.1** *(Stability) The discretization (6.10) is stable i.e.*

$$\|U^n\|_\infty \leq \max(\|U^0\|_\infty, C_b).\tag{6.14}$$

**Proof.** The discrete equations are

$$(1 + \Delta t_{n+1}(\alpha_i^{n+1} + \beta_i^{n+1} + r))u_i^{n+1} = \bar{u}_i^n + \Delta t_{n+1}\alpha_i^{n+1}u_{i-1}^{n+1} + \Delta t_{n+1}\beta_i^{n+1}u_{i+1}^{n+1}.$$

Since we take  $\bar{u}_i^n$  as linear interpolation of  $u_i^n$  and  $u_{i+1}^n$  in the discretization, there holds

$$(1 + \Delta t_{n+1}(\alpha_i^{n+1} + \beta_i^{n+1} + r))|u_i^{n+1}| \leq \|U^n\|_\infty + \Delta t_{n+1}(\alpha_i^{n+1} + \beta_i^{n+1})\|U^{n+1}\|_\infty.$$

If  $\|U^{n+1}\|_\infty = |u_j^{n+1}|$ ,  $0 < j < N$ , then we have

$$(1 + \Delta t_{n+1}(\alpha_j^{n+1} + \beta_j^{n+1} + r))\|U^{n+1}\|_\infty \leq \|U^n\|_\infty + \Delta t_{n+1}(\alpha_j^{n+1} + \beta_j^{n+1})\|U^{n+1}\|_\infty,$$

which implies that

$$\|U^{n+1}\|_\infty \leq \|U^n\|_\infty.$$

If  $j = 0$  or  $j = N$ , then  $\|U^{n+1}\|_\infty = |u_0^{n+1}| \leq |u_0^n|$  or  $\|U^{n+1}\|_\infty = |u_N^{n+1}| \leq C_b$ .

Thus, we have

$$\|U^{n+1}\|_\infty \leq \max(\|U^0\|_\infty, C_b)$$

which complete the prove.  $\square$

**Lemma 6.2** (Consistency) *For any smooth function  $v$  with  $v_i^n = v(t^n, x_i)$ , the discrete scheme (6.10) is consistent.*

**Proof.** Using Taylor's expansion, we have

$$|(\frac{1}{2}\sigma\partial_{xx}u - rv)_i^n - (L_h^\sigma v^n)_i| = O(\Delta x),$$

and using expansion along characteristic direction  $c$

$$\begin{aligned} & |(\Psi \frac{\partial v}{\partial c})_i^{n+1} - \frac{v_i^{n+1} - \bar{v}_i^n}{\Delta t_{n+1}}| \\ &= O(\Delta x + \Delta t). \end{aligned}$$

where  $\bar{v}_i^n = v(t^n, \bar{x}_i^n)$ .

For smooth  $v$ , by Taylor's expansion, we can derive the discretization error as follows

$$\begin{aligned} & \left| (\Psi \frac{\partial v}{\partial \tau})_i^{n+1} - \sup_{\sigma \in \{\underline{\sigma}, \bar{\sigma}\}} (L^\sigma v^{n+1})_i - \left[ \frac{v_i^{n+1} - \bar{v}_i^n}{\Delta t_{n+1}} - \sup_{\sigma^{n+1} \in \{\underline{\sigma}, \bar{\sigma}\}} (L_h^{\sigma^{n+1}} v^{n+1})_i \right] \right| \\ & \leq \left| (\Psi \frac{\partial v}{\partial \tau})_i^{n+1} - \frac{v_i^{n+1} - \bar{v}_i^n}{\Delta t_{n+1}} \right| + \sup_{\sigma \in \{\underline{\sigma}, \bar{\sigma}\}} |(L^\sigma v^{n+1})_i - (L_h^\sigma v^{n+1})_i| \\ & = O(\Delta t + \Delta x) \end{aligned}$$

which prove the consistency of the discretization schem.  $\square$

**Lemma 6.3** (Monotonicity) *The discretization (6.10) is monotone.*



**Proof.** For  $i = 0$  or  $i = N$  the Lemma is trivially true. For  $0 < i < N$ , we write equation (6.12) in component form

$$\begin{aligned} & G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) \\ = & \frac{u_i^{n+1} - u_i^n}{\Delta t_{n+1}} - \frac{\bar{u}_i^n - u_i^n}{\Delta t_{n+1}} \\ & - \sup_{\sigma^{n+1} \in [\underline{\sigma}, \bar{\sigma}]} [\alpha_i^{n+1}(\sigma_i^{n+1})u_{i-1}^{n+1} + \beta_i^{n+1}(\sigma_i^{n+1})u_{i+1}^{n+1} - (\alpha_i^{n+1}(\sigma_i^{n+1}) + \beta_i^{n+1}(\sigma_i^{n+1}) + r)u_i^{n+1}]. \end{aligned}$$

For  $\varepsilon \geq 0$ , we have

$$\begin{aligned} & G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1} + \varepsilon, \bar{u}_i^n) - G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) \\ \leq & \sup_{\sigma^{n+1} \in [\underline{\sigma}, \bar{\sigma}]} \{-\beta_i^{n+1}(\sigma_i^{n+1})\varepsilon\} = -\varepsilon \inf_{\sigma^{n+1} \in [\underline{\sigma}, \bar{\sigma}]} \{\beta_i^{n+1}(\sigma_i^{n+1})\} \leq 0. \end{aligned}$$

With the similar argument we derive

$$G_i^{n+1}(h, u_{i-1}^{n+1} + \varepsilon, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) - G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) \leq 0.$$

It is easy to check that

$$G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n + \varepsilon) - G_i^{n+1}(h, u_{i-1}^{n+1}, u_i^{n+1}, u_{i+1}^{n+1}, \bar{u}_i^n) = -\frac{\varepsilon}{\Delta t_{n+1}} \leq 0.$$

Thus we prove the discrete scheme (6.10) is monotone.  $\square$

The discrete scheme (6.10) is consistent, stable and monotone, from [5], we have the following convergence theorem

**Theorem 6.2** (Convergence to the viscosity solution) *The solution of the discrete scheme (6.10) converges to the viscosity solution of equation (6.1).*

## 6.2 Iterativ Solution of Discrete Algebraic System

In the previous subsection, we have show that the solution of the discretization (6.7) converges to the viscosity solution of the nonlinear PDE (6.1). Since the implicit scheme leads a nonlinear algebraic system (6.10) at each timestep, the discretization is not a practical scheme. In this section, we aim to solve this discrete scheme by a practical iterative method.

Iterative Algorithm for (6.10)

1.  $n = 0$
2. Set  $k = 0$  and  $\tilde{u}^k = U^n$
3. For  $k = 0, 1, 2, \dots$

Solve

$$\begin{aligned} & [I - \Delta t_{n+1} A^{n+1}(\tilde{\sigma}^k)] \tilde{u}^{k+1} = \tilde{U}^n \\ & \tilde{\sigma}_i^k \in \operatorname{argsup}_{\sigma \in [\underline{\sigma}, \bar{\sigma}]} \{[A^{n+1}(\sigma) \tilde{u}^k]_i\} \end{aligned} \quad (6.15)$$

4. If  $\max_i \frac{|\tilde{u}_i^{k+1} - \tilde{u}_i^k|}{\max(\text{scale}, \tilde{u}_i^{k+1})} < \text{tolerance}$  then quit, else  $k = k + 1$  go to 3.
5. Set  $U^{n+1} = \tilde{u}^{k+1}$ ,  $\hat{\sigma}^{n+1} = \tilde{\sigma}^k$  and  $n = n + 1$ , go to 2.

The term *scale* is used to ensure that unrealistic levels of accuracy are not required when the value is very small. In the iterative algorithm  $\tilde{\sigma}^k$  is given by

$$\tilde{\sigma}_i^k = \begin{cases} \bar{\sigma}, & \text{if } \frac{(\tilde{u}_{i+1}^k - \tilde{u}_i^k)/(x_{i+1} - x_i) - (\tilde{u}_i^k - \tilde{u}_{i-1}^k)/(x_i - x_{i-1})}{x_{i+1} - x_{i-1}} \geq 0 \\ \underline{\sigma}, & \text{if } \frac{(\tilde{u}_{i+1}^k - \tilde{u}_i^k)/(x_{i+1} - x_i) - (\tilde{u}_i^k - \tilde{u}_{i-1}^k)/(x_i - x_{i-1})}{x_{i+1} - x_{i-1}} < 0 \end{cases}.$$

**Theorem 6.3** (Convergence of the Iterative Algorithm) The iteration algorithm (6.15) for (6.10) converges to the unique solution of equation (6.7) for any initial iterate  $\tilde{u}^0$ .

**Proof.** First, we will prove that  $\|\tilde{u}^k\|_\infty$  is bounded independent of iteration  $k$  with a similar argument we use in Lemma 6.1.

We can write (6.15) in component form as follows

$$(1 + \Delta t_{n+1}(\alpha_i^{n+1}(\tilde{\sigma}^k) + \beta_i^{n+1}(\tilde{\sigma}^k) + r))\tilde{u}_i^{k+1} - \Delta t_{n+1}\alpha_i^{n+1}(\tilde{\sigma}^k)\tilde{u}_{i-1}^{k+1} - \Delta t_{n+1}\beta_i^{n+1}(\tilde{\sigma}^k)\tilde{u}_{i+1}^{k+1} = \tilde{u}_i^n.$$

Then

$$\begin{aligned} & (1 + \Delta t_{n+1}(\alpha_i^{n+1}(\tilde{\sigma}^k) + \beta_i^{n+1}(\tilde{\sigma}^k) + r))|\tilde{u}_i^{k+1}| \\ & \leq |\tilde{u}^n| + |\Delta t_{n+1}\alpha_i^{n+1}(\tilde{\sigma}^k)\tilde{u}_{i-1}^{k+1} + \Delta t_{n+1}\beta_i^{n+1}(\tilde{\sigma}^k)\tilde{u}_{i+1}^{k+1}| \\ & \leq \|U^n\|_\infty + \Delta t_{n+1}(\alpha_i^{n+1}(\tilde{\sigma}^k) + \beta_i^{n+1}(\tilde{\sigma}^k))\|\tilde{u}^{k+1}\|_\infty. \end{aligned}$$

Thus

$$\|\tilde{u}^{k+1}\|_\infty \leq \|U^n\|_\infty \leq \max(\|U^0\|_\infty, C_b),$$

which means that  $\|\tilde{u}^{k+1}\|_\infty$  is bounded independent of  $k$ .

Now we will prove that the iterates  $\{\tilde{u}^k\}$  form a nondecreasing sequence. From (6.15), the iterates difference  $\tilde{u}^{k+1} - \tilde{u}^k$  satisfy

$$[I - \Delta t_{n+1}A^{n+1}(\tilde{\sigma}^k)](\tilde{u}^{k+1} - \tilde{u}^k) = \Delta t_{n+1}[A^{n+1}(\tilde{\sigma}^k) - A^{n+1}(\tilde{\sigma}^{k-1})]\tilde{u}^k. \quad (6.16)$$

Notice that

$$\tilde{\sigma}_i^k \in \operatorname{argmax}_{\sigma \in \{\underline{\sigma}, \bar{\sigma}\}} \{[A^{n+1}(\sigma)]\tilde{u}^k\}_i,$$

the right hand side of (6.16) is nonnegative, i.e.

$$\Delta t_{n+1}\{[A^{n+1}(\tilde{\sigma}^k) - A^{n+1}(\tilde{\sigma}^{k-1})]\tilde{u}^k\}_i \geq 0.$$

Consequently,

$$[(I - \Delta t_{n+1}A^{n+1}(\tilde{\sigma}^k))(\tilde{u}^{k+1} - \tilde{u}^k)]_i \geq 0. \quad (6.17)$$

From Theorem 6.1 we know that matrix  $I - \Delta t_{n+1}A^{n+1}(\tilde{\sigma}^k)$  is an M-matrix and hence

$$[I - \Delta t_{n+1}A^{n+1}(\tilde{\sigma}^k)]^{-1} \geq 0. \quad (6.18)$$

From (6.17) (6.18), we can derive that

$$\tilde{u}^{k+1} - \tilde{u}^k \geq 0,$$

which prove that the iterates form a nondecreasing sequence. The iterates sequence  $\{\tilde{u}^k\}$  is nondecreasing and bounded, thus the sequence converges to a solution, i.e.,  $\exists \tilde{u}$  such that  $\|\tilde{u}^k - \tilde{u}\|_\infty \rightarrow 0$  and

$$\begin{aligned} & [(I - \Delta t_{n+1}A^{n+1}(\tilde{\sigma}))\tilde{u} = \tilde{u}^n \\ & \tilde{\sigma}_i \in \operatorname{argsup}_{\sigma \in \{\underline{\sigma}, \bar{\sigma}\}} \{[A^{n+1}(\sigma)]\tilde{u}\}_i \end{aligned} \quad (6.19)$$

Since the matrix  $I - \Delta t_{n+1}A^{n+1}(\tilde{\sigma})$  is a M-matrix, thus the solution of (6.19) is unique which finish the proof.  $\square$

In next section we will simulate the ask (resp. bid) price of the contingent claim by using monotone characteristic finite difference schemes for (5.21) (resp. (5.22)). The convergence of the solution of the monotone characteristic finite difference schemes (6.7) to the viscosity solution of (6.1) guarantee the simulation ask (resp. bid) price of the contingent claim convergence to the correct financial relevant solution.

## 7 Examples and simulations

In this section, we will give simulations for the bid-ask pricing mechanisms of contingent claims under uncertainty with payoff given by some function  $\phi(S_T)$ . In computer simulations, we only make the numerical program for the nonlinear PDE (5.21), since the bid price  $u^b(t, x) := -u(t, x)$  where  $u(t, x)$  is the viscosity solution of (5.21) with terminal condition  $u(t, x) = -\phi(x)$ .

**Example 7.1** *Digital call option under uncertain volatility.*

We consider a digital call option with the payoff as follows

$$\phi(S_T) = \begin{cases} 1, & S_T \geq K \\ 0, & S_T < K \end{cases}.$$

The strike price of the digital option is  $K = 100$  and the maturity is six months  $T = 0.5$ . The volatility bounds are given by  $\underline{\sigma} = 0.15, \bar{\sigma} = 0.25$  and short interest rate is  $r = 0.10$ .

We use the numerical schemes constructed in Section 3 to compute the nonlinear PDE (5.21) with the payoff function as initial condition and the boundary condition is  $\phi(S_{\max}) = 1$ . we choose the grid as  $\Delta s = 1, \Delta t = 0.0025$  and iterative tolerance  $= 10^{-6}$ . We plot Fig. 1 the ask (top left) price and bid (top right) price surfaces of the digital call option.

**Example 7.2** *Butterfly option under uncertain volatility.*

The second example is a butterfly option with the payoff as follows

$$\phi(S_T) = \max(S - K_1, 0) - 2\max(S - (K_1 + K_2)/2, 0) + \max(S - K_2, 0),$$

the boundary condition is  $u^a(S_{\max}) = 0, K_1 = 90$  and  $K_2 = 110$ . The other parameters used here are the same as that we used in example 1. The ask price (down left) and the bid price (down right) surfaces of the butterfly option are shown in Fig. 1.

Fig. 1 shows that the ask price surface is above the bid price surface for the both claims, and the ask (resp. bid) price dynamic keeps the monotone intervals and convex (resp. concave) intervals of the corresponding payoff function which verify the theoretical results we showed in Lemma 5.1.

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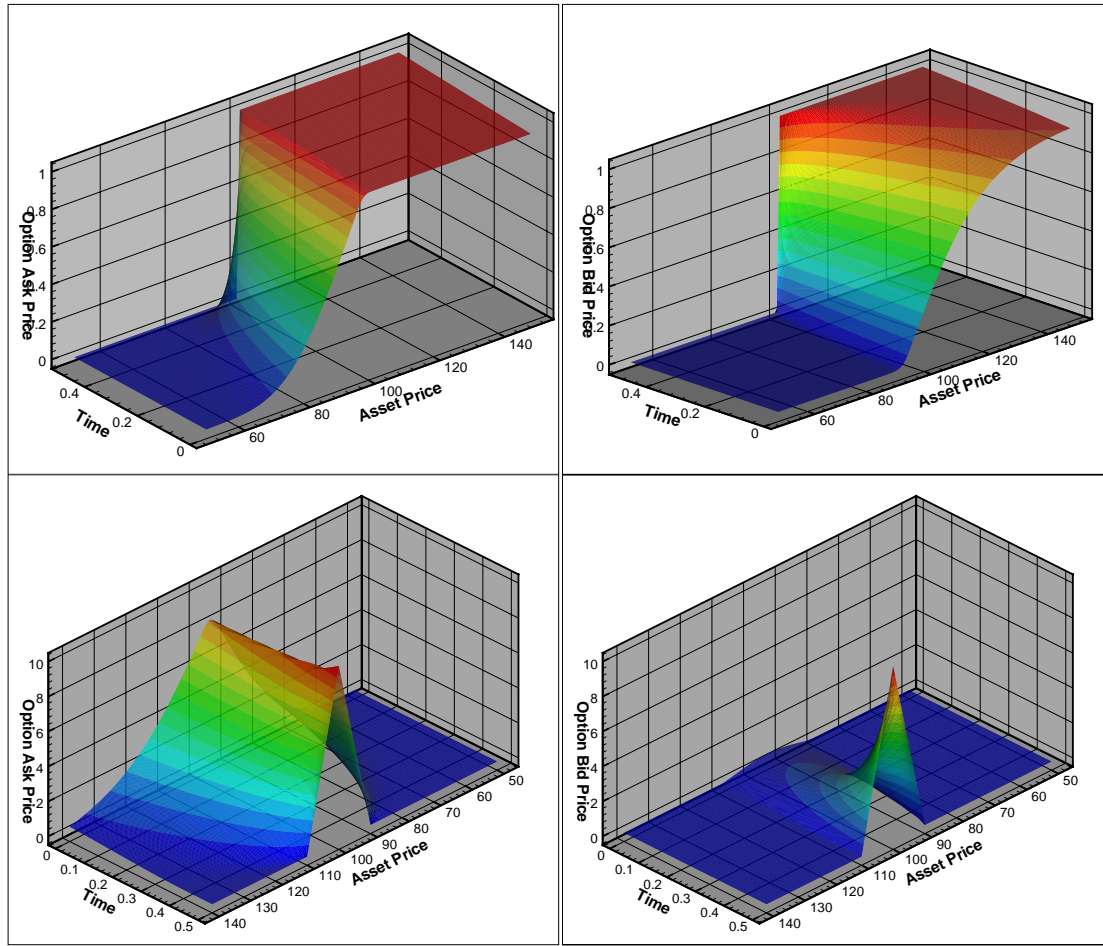


Figure 1: The ask (left) and the bid (right) price surfaces of the digital option (up) and the butterfly option (down).

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